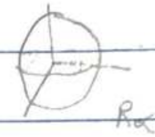
10.22

Last time: More Double Integration

Q: What is the volume of a sphere of radius $\alpha > 0$?

Sol: (w/ what we know)

picture



$$Vol(S_\alpha) = \iint_{R_\alpha} h(x,y)\, dA$$

$x^2+y^2+z^2=\alpha^2$

The sphere with the equation $x^2+y^2+z^2=\alpha^2$ has an upper hemisphere w/ equation $z=\sqrt{\alpha^2-x^2-y^2}$ and the lower hemisphere w/ an equation $z=-\sqrt{\alpha^2-x^2-y^2}$

"Height function = (upper hemisphere) - (lower hemisphere)"

$$h(x,y) = \sqrt{\alpha^2-x^2-y^2} - \left(-\sqrt{\alpha^2-x^2-y^2}\right) = 2\sqrt{\alpha^2-x^2-y^2}$$

Now, the region of integration is:

$$R = \{(x,y) : x^2+y^2 \le \alpha^2\}$$

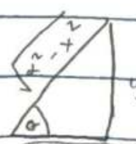
Now the upper semi-circle boundary R_α is $y=\sqrt{\alpha^2-x^2}$ and lower semi-circle is $y=-\sqrt{\alpha^2-x^2}$

$$R_\alpha = \{(x,y) : -\alpha \le x \le \alpha \quad -\sqrt{\alpha^2-x^2} \le y \le \sqrt{\alpha^2-x^2}$$

Hence, $$Vol(S_\alpha) = \int_{x=-\alpha}^{\alpha} \int_{\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}} 2\sqrt{\alpha^2-x^2-y^2}\, dy\, dx$$

Inner Int

$$\int_{y=-\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}} 2\sqrt{(\alpha^2-x^2)-y^2}\, dy$$



Indefinitely $\int 2\sqrt{\alpha^2-x^2-y^2}\, dy$

$$= 2\int \sqrt{\alpha^2-x^2}\cos\theta \cdot \sqrt{\alpha^2-x^2}\cos\theta\, d\theta$$

$$= 2(\alpha^2-x^2)\int \cos^2\theta\, d\theta$$

$$= (\alpha^2-x^2)\int (1+\cos(2\theta))\, d\theta$$

$$= (\alpha^2-x^2)\left(\theta + \tfrac{1}{2}\sin(2\theta)\right) + C$$

$$= (\alpha^2-x^2)\left(\theta + \sin\theta\cos\theta\right) + C$$

$\sin(\theta) = \frac{y}{\sqrt{\alpha^2-x^2}}$

$y = \sqrt{\alpha^2-x^2}(\sin\theta)$

$dy = \sqrt{\alpha^2-x^2}(\cos\theta)\, d\theta$

$\sqrt{\alpha^2-x^2-y^2} = \sqrt{\alpha^2-x^2}(\cos\theta)$

$$=(\alpha^2-x^2)\left(\arcsin\left(\frac{y}{\sqrt{\alpha^2-y^2}}\right)+\frac{y}{\sqrt{\alpha^2-x^2}}\cdot\frac{\sqrt{\alpha^2-x^2-y^2}}{\sqrt{\alpha^2-x^2}}\right)+C$$

$$=(\alpha^2-x^2)\arcsin\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)+y\sqrt{\alpha^2-x^2-y^2}+C$$

$$\int_{y=-\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}} 2\sqrt{\alpha^2-x^2-y^2}\,dy$$

$$=\left[(\alpha^2-x^2)\arcsin\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)+y\sqrt{\alpha^2-x^2-y^2}\right]_{-\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}}$$

$$=\left((\alpha^2-x^2)\arcsin(1)+\sqrt{\alpha^2-x^2}\sqrt{0}\right)-\left((\alpha^2-y^2)\arcsin(-1)-\sqrt{\alpha^2-x^2}\sqrt{0}\right)$$

$$=(\alpha^2-x^2)(\arcsin(1)-\arcsin(-1))=(\alpha^2-x^2)\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\boxed{\pi(\alpha^2-x^2)}$$

Outer Int

$$\int_{x=-\alpha}^{\alpha}\pi(\alpha^2-x^2)\,dx$$

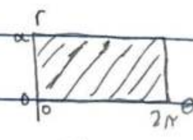
$$=\pi\left(\alpha^2x-\frac{1}{3}x^3\right)\Big|_{-\alpha}^{\alpha}$$

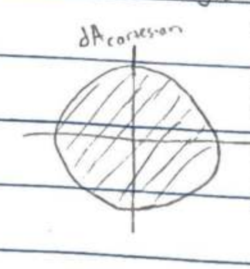
$$=\pi\left(\left(\alpha^3-\frac{1}{3}\alpha^3\right)-\left(\alpha^3+\frac{1}{3}\alpha^3\right)\right)$$

$$=\pi\alpha^3\left(2-\frac{2}{3}\right)=\boxed{\frac{4}{3}\pi\alpha^3}\leftarrow \mathrm{Vol}(S_\alpha)$$

NB: This computation above was complicated. It seems it would be more natural to use polar coordinates to describe both the region and height function.

The region: ⟶





cartesian $\boxed{x^2+y^2\le\alpha^2}$

$\boxed{0\le r\le\alpha}$

$\boxed{0\le\theta\le 2\pi}$ polar

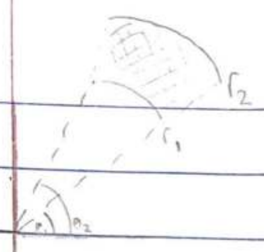
Similarly the height function:

$h(r\cos\theta, r\sin\theta)=2\sqrt{\alpha^2-r^2}$

Need: How is the differential affected?

Want: Formula for $dA_{cartesian}$ in terms of dA_{polar}

(Think abt a circular sector)



From high School Geometry: Area of full circle section is $\frac{1}{2}\theta \cdot r^2$

The circular sector corresponding to polar rectange $[\theta_1, \theta_2] \times [r_1, r_2]$ has area $\frac{1}{2}(\theta_2 - \theta_1) r_2^2 - \frac{1}{2}(\theta_2 - \theta_1) r_1^2$

$$= \frac{1}{2}(\theta_2 - \theta_1)(r_2^2 - r_1^2)$$

$$= \frac{1}{2}(r_1 + r_2)(\theta_2 - \theta_1)(r_2 - r_1)$$

$\therefore$ The corresponding areas are

$$\Delta A_{cart} = \frac{1}{2}(r_1 + r_2)\, \Delta\theta\, \Delta r = \frac{1}{2}(r_1 + r_2)\, \Delta A_{polar}$$

Now limiting the $\Delta A_{polar} \to 0$ (via $\Delta\theta \to 0$ and $\Delta r \to 0$)
we see that $\frac{1}{2}(r_1 + r_2) = \frac{1}{2}(2r_2 - \Delta r) = r_2 + \frac{1}{2}\Delta r \longrightarrow r$

So in the limit $dA_{cart} = r\, dA_{polar}$

Volume of sphere, take 2

Sol: (via Polar Coordinates)

$$Vol(S_\alpha) = \iint_{R_{cart}} h(x, y)\, dA_{cart}$$

$$= \iint_{R_{polar}} h(r\cos\theta, r\sin\theta)\, r\, dA_{polar}$$

Go back & look at sphere ↑

$z^2 = \alpha^2 - r^2$

$Z_{upper} = \sqrt{\alpha^2 - r^2}$

$Z_{lower} = \sqrt{\alpha^2 - r^2}$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\alpha} 2\sqrt{\alpha^2 - r^2}\; r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\alpha} -u^{1/2}\, du\, d\theta$$

Usub

$$= -\int_{\theta=0}^{2\pi} \left[\frac{2}{3} u^{3/2}\right]\Big|_{0}^{\alpha} d\theta$$

$$= -\int_{\theta=0}^{2\pi} \left[\frac{2}{3}(\alpha^2 - r^2)^{3/2}\right]_{r=0}^{\alpha} d\theta$$

$$= -\frac{2}{3} \int_{\theta=0}^{2\pi} \left((\alpha^2 - x^2)^{3/2} - (\alpha^2 - 0^2)^{3/2} \right) d\theta$$

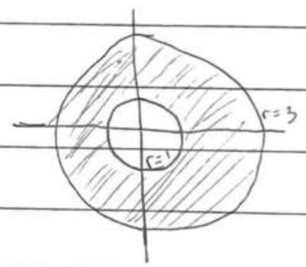
$$= -\frac{2}{3} - \alpha^3 \int_{\theta=0}^{2\pi} d\theta$$

$$= \frac{2}{3} \alpha^3 [\theta]_{\theta=0}^{2\pi}$$

$$\boxed{= \frac{4}{3} \pi \alpha^3}$$

Ex: Compute $\iint_R \cos\left(\sqrt{x^2+y^2}\right) \, dA$ for R the annulus between $x^2+y^2=1$ and $x^2+y^2=9$

Solution: Set this up using polar coordinates:



$$\cos\left(\sqrt{x^2+y^2}\right) = \cos\left(\sqrt{r^2}\right) = \cos(r) \quad (\text{b/c } r \geq 0)$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\therefore \iint_{R_{rect}} \cos\left(\sqrt{x^2+y^2}\right) dA_{rect} = \iint_{R_{polar}} \cos(r) \, r \, dA_{polar}$$

$R_{polar} = \{(r,\theta) : 1 \leq r \leq 3, \ 0 \leq \theta \leq 2\pi$

$$= \int_{r=1}^{3} \int_{\theta=0}^{2\pi} r\cos(r) \, d\theta \, dr$$

$$= \int_{r=1}^{3} r\cos(r) \, [\theta]_{\theta=0}^{2\pi} \, dr$$

$$\begin{pmatrix} u = r & dv = \cos(r)\,dr \\ du = dr & v = \sin(r) \end{pmatrix}$$

$$= 2\pi \int_{r=1}^{3} r\cos(r) \, dr$$

$$= 2\pi \left[r\sin r - \int \sin(r) \, dr \right]_{r=1}^{3}$$

$$= 2\pi \left[r\sin(r) + \cos(r) \right]_{r=1}^{3}$$

$$= 2\pi \left((3\sin(3) + \cos(3)) - (\sin(1) + \cos(1)) \right)$$

Excersise: Compute $\iint_R y\exp(-x^2-y^2)\, dA$ on the region R the quarter annulus in the first quadrant. Between $x^2+y^2=25$ and $x^2+y^2=4$.